[3] Koval, L. R., "Structural responses to dynamic loading, Part I—Stability of axially-loaded thin cylindrical shells under dynamic lateral pressures," AIAA Preprint 64-485 (June 29–July 2, 1964).

[4] Trevana, D. H., *Static Fields in Electricity and Magnetism* (Butterworths Scientific Publications Ltd., London, 1961), pp. 5, 101, 102–103, 120.

[5] Shire, E. S., *Classical Electricity and Magnetism* (Cambridge University Press, Cambridge, England, 1960), pp. 22, 68–71.

Newtonian Entropy Layer in the Vicinity of a Conical Symmetry Plane

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In their book,[1] Hayes and Probstein considered the Newtonian theory for the flow field in the vicinity of a conical-symmetry plane. It is well known[2, 3] that Newtonian theory is not uniformly valid near the body surface where an entropy layer forms. In a recent analysis to be published shortly,[4] this author applied a generalization of the method of inner and outer expansions to the hypersonic flow over a general conical surface and obtained a thin shock-layer theory that is uniformly valid in the entropy layer. It is the purpose of this note to apply this general theory to the conical symmetry plane problem so as to illuminate the general nature of the entropy layer corrections to the basic Newtonian theory.

Outer Solution (Newtonian Theory)

The general conical problem is formulated in conical-curvilinear coordinates r, ξ, η, where r is the radial coordinate and ξ, η is a set of orthogonal-curvilinear coordinates on the spherical surface r = const (see Fig. 1 for notation and convention). In the formal asymptotic analysis of Ref. 4, the first term of the outer expansion is completely equivalent to the usual Newtonian theory. Since the analytic properties of the Newtonian theory for general conical surfaces have been described in some detail in Ref. 5, we will not consider the general theory herein. We simply note, for later comparison with the inner solution, that the appropriate scalings for the outer region are given by $\bar{\eta} = \epsilon^{-1}\eta$, $\bar{v}_\eta = \epsilon^{-1}v_\eta/V_\infty$, $\bar{\rho} = \epsilon\rho/\rho_\infty$, where η is the coordinate, v_η is the velocity component normal to the body surface, ρ is the density, V_∞ is the freestream velocity, ρ_∞ is the freestream density, and ϵ is the small parameter that in the present problem will be set equal to the density ratio in the plane of symmetry. The radial velocity v_r, the lateral velocity v_ξ, the pressure p, and the entropy S, are 0(1) with respect to ϵ, and the other coordinate ξ is left unstretched.

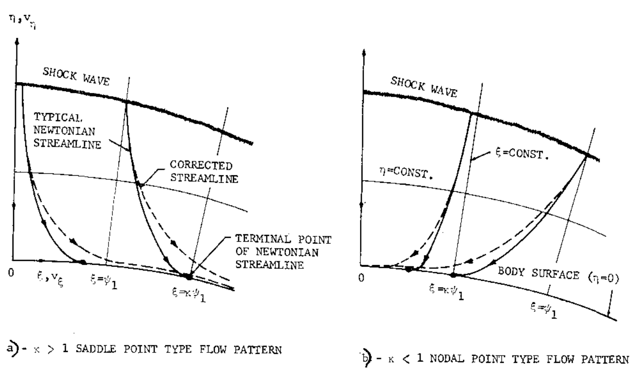

Fig. 1 Streamline geometry near a symmetry plane.

The main deviation from the Newtonian solution is centered about the solution for the crossflow streamlines. Hayes and Probstein do not carry the Newtonian solution through to obtain the equation for the streamline near the symmetry plane. However, this result is given by Eq. (6.119) in Ref. 5, and when converted to the Hayes-Probstein notation, it reads

$$\eta = \epsilon \tan\sigma_0[\kappa(\kappa - 1)^{-2}(\xi/\kappa\psi - 1 - \ln\xi/\kappa\psi) + 0(\xi^2)] \quad (1)$$

where σ_0 is the flow deflection angle in the symmetry plane and κ is the ratio of the surface curvature in the symmetry plane to the curvature of a circular cone of half angle σ_0. The crossflow stream function ψ is defined so that $\psi = \xi$ at the shock wave, and $\xi = x_1/y_1$ where x_1 and y_1 are the Cartesian coordinates appearing in the Hayes-Probstein solution. Thus,

$$\eta_s = \epsilon \tan\sigma_0 \left[\frac{\kappa \ln\kappa - (\kappa - 1)}{(\kappa - 1)^2} + 0(\xi^2)\right] \quad (2)$$

In Fig. 1 we have sketched a number of typical Newtonian streamlines for the case $\kappa > 1$ and for $\kappa < 1$. We observe that a typical streamline ψ enters the shock layer at a station $\xi = \psi$ and terminates at the body surface at station $\xi = \kappa\psi$. We note that for $\kappa > 1$, the streamline pattern has a saddle point and for $\kappa < 1$ it has a nodal-point character. For $\kappa = 1$, there is no crossflow and the streamlines lie along body normals.

The Newtonian streamlines are tangent to the body surface at the termination point and, therefore, the body surface is an envelope of Newtonian streamlines. If we also require the body surface to be a stream surface, which it must be if the lateral pressure gradient is nonzero, it is clear that the stream function ψ must be discontinuous at the body surface. It follows that S, v_r, and ρ are also discontinuous at the body surface in the Newtonian approximation.

It is important to note that the point 0 in Fig. 1 is not an isolated stagnation point within the Newtonian approximation. As a result, we find that this point is neither a branch-point nor a nodal-point singularity of the Newtonian equations. On the contrary, we find that these expected point singularities are spread out over the body surface as $\epsilon \to 0$.

Inner Solution (Entropy Layer)

In the previous paragraphs we have seen that the Newtonian solution is discontinuous at the body surface. If we try to improve the solution by constructing higher-order terms of the outer expansion, we find that these additional terms are logarithmically singular[2–4] at the body surface. A preliminary analysis indicates that this breakdown of the outer expansion is caused by the neglect of the effect of the lateral pressure gradients on the lateral component of velocity, v_ξ. This effect is accounted for in a systematic fashion by constructing a formal asymptotic expansion that, by design, is valid near the body surface.

By a relatively simple order of magnitude analysis we find that the scaled variables appropriate for the inner region are given by

$$\tilde{\rho} = \bar{\rho} \qquad \tilde{v}_\xi = \epsilon^{-1}\bar{v}_\xi = \epsilon^{-1}v_\xi/V_\infty$$

$$\tilde{v}_\eta = -\bar{v}_\eta/\bar{\eta} \qquad \tilde{\eta} = -\epsilon\ln[\bar{\eta}/F(\psi)]$$

where F is an arbitrary function introduced in order to simplify some of the final formulas. The remaining variables are unscaled with respect to ϵ. The inner solution is obtained by expanding the exact equations, written in terms of inner variables in powers of ϵ. It turns out that the pressure is constant across the inner region to all orders. The solution for the remaining variables can be reduced to quadrature

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by carrying out the expansion in Von Mises streamline coordinates ξ, ψ, as in the outer region. The solution of the first-order inner equations for a general conical surface is given below[4]:

$$v_r = R(\psi) \qquad v_{\tilde{\eta}} = -2R(\psi)\tilde{\eta}$$
$$v_\xi \ (1/\rho R)(dp_b/d\xi) \qquad p = p_b(\xi)$$
$$S = S(\psi) \qquad \rho = \rho(p_b,S) \tag{3}$$
$$\tilde{\eta} = -2\int_{\xi^*(\psi)}^{\xi} \epsilon\rho R^2 \left(\frac{dp_b(t)}{dt}\right)^{-1} dt$$

where R, S, and ξ^* are arbitrary functions of ψ, and $p_b(\xi)$ is the surface pressure. All of these functions are to be determined along with $F(\psi)$ by matching conditions. It is important to note that the functions ξ^* and F are not independent. One can be selected arbitrarily and the remaining one is fixed by the matching condition.

In this problem, the inner and outer regions unfortunately do not overlap and the expansions cannot be matched as they stand. Usually, the thing to do is to introduce a third expansion that is valid in some transition region that overlaps the inner and outer regions. The author has found it convenient to proceed via an alternative approach which enables one to resolve the matching problem without introducing a third expansion. Basically, the idea is to apply the Poincaré-Lighthill-Kuo (PLK) method[6] to the outer expansion to weaken the singularities present in that expansion, thereby extending its region of validity. This amounts to replacing the variable ξ in the Newtonian solution by the "strained" coordinate z, which is defined by the expansion

$$\xi = z + \epsilon\xi^{(1)}(z, \psi) + \ldots \tag{4}$$

where $\xi^{(1)}(z, \psi)$ is to be determined by the condition that the regions of validity of the strained outer solution and inner solution overlap. It can be shown[4] that this condition is satisfied by the choice

$$\xi^{(1)}(z, \psi) = \left[\frac{\epsilon^{-1}}{\rho(p_b,S)R^2}\frac{dp_b(\xi)}{d\xi}\right]_{\xi=\xi_b} \times \ln\frac{\xi_b - z}{\xi_b - \psi} + \xi_0{}^1(\psi)\left[1 - \frac{\xi_b - z}{\xi_b - \psi}\right] \tag{5}$$

where $\xi_0{}^{(1)}(\psi)$ is a regular function of ψ that will not enter into the first-order uniformly valid solution, and $\xi_b(\psi)$ is given by

$$\xi_b(\psi) = \psi + \tan^{-1}[v_{\xi\infty}(\psi)/v_{r\infty}(\psi)] \tag{6}$$

By carrying out the matching, we find that $p_b(\xi)$ and $S(\psi)$ are given by their Newtonian values and $R(\psi)$, $\xi^*(\psi)$, and $F(\psi)$ are given by the relations

$$\xi^*(\psi) = \xi_b(\psi) \qquad R(\psi) = [v_{\xi\infty}{}^2(\psi) + v_{r\infty}{}^2(\psi)]^{1/2}$$
$$F(\psi) = -\frac{1}{2}\left[\frac{d\xi_b}{d\psi}\frac{\rho_\infty\epsilon^{-1}}{\rho(p_b,S)}\right]^{-1}_{\xi=\xi_b}\left[\frac{v_{\eta\infty}(\psi)}{v_{\xi\infty}(\psi)}\right] \times \left[1 + \frac{v_{r\infty}{}^2(\psi)}{v_{\xi\infty}{}^2(\psi)}\right]^{1/2}(\xi_b - \psi)^2 \tag{7}$$

where the freestream velocity components are to be evaluated at the body surface. A composite expansion can be constructed in the usual way.

Uniformly Valid Solution at a Symmetry Plane

A solution near a symmetry plane can be obtained by replacing ξ by z in Eq. (1) and then expanding $\xi^{(1)}(z, \psi)$ and the inner solution in powers of ξ. Carrying this out for the streamlines we find the following:

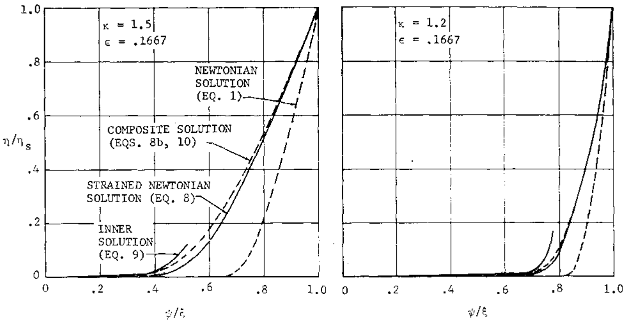

Fig. 2 Stream-function distribution near a saddle point.

Outer solution

$$\eta(z, \psi) = \epsilon\tan\sigma_0\,[\kappa(\kappa - 1)^{-2}(z/\kappa\psi - 1 - \ln z/\kappa\psi)] \tag{8a}$$

$$\xi(z, \psi) = z + \epsilon\kappa\psi g(\kappa)\ln[(\kappa/\kappa - 1)(1 - z/\kappa\psi)] \tag{8b}$$

where

$$g(\kappa) = -3[\kappa/\kappa - 1]^2[(\kappa - 3)(\kappa - 1) + 2\ln\kappa] \tag{8c}$$

Inner solution

$$\eta(\xi, \psi) = \epsilon N(\psi)[\xi/\kappa\psi]^{\mu(\psi)/\epsilon} \tag{9}$$

where for $\kappa > 1$ (saddle point), $N(\psi) = \tan\sigma_0/2\kappa$, $\mu(\psi) = 2/g(\kappa)$; and for $\kappa < 1$ (nodal point),

$$\mu(\psi) = 2\left(\frac{\epsilon^{-1}}{\rho(p_b,S)R^2}\frac{d^2p_b}{d\xi^2}\right)^{-1}_{\xi=0}$$

and $N(\psi)$ is a function that depends on the stream-function distribution away from the nodal point. Because the streamlines at the base of a nodal point cross the shock wave at a distance from the symmetry plane, the functions $N(\psi)$, $S(\psi)$, and $\mu(\psi)$ are not determined by local conditions near the symmetry plane. For a saddle point, the solution is entirely local, and we can carry the explicit solution one step further and construct the composite solution for the streamlines. The result is given below where ($\kappa > 1$, saddle point):

Composite solution

$$\eta(\xi, z, \psi) = \epsilon[\xi/z]^{2/\epsilon g(\kappa)}[\kappa\tan\sigma_0/(\kappa - 1)^2] \times \left[\frac{z}{\kappa\psi} - 1 - \ln\left(\frac{z}{\kappa\psi}\right)\right]\left[\frac{1/\kappa - 1}{(z/\kappa\psi) - 1}\right]^{2\kappa\psi/z} \tag{10}$$

Equations (10) and (8b) yield a parametric representation for the uniformly valid solution for the streamlines near the plane of symmetry of a saddle point.

In Fig. 2, we have plotted the stream-function distribution across the shock layer in the vicinity of a saddle-point plane of symmetry for $\epsilon = 0.1667$ and $\kappa = 1.5$, 1.2. These results clearly show the very strong boundary-layer effects near the cone surface. We note the very large effect of the PLK straining on the outer solution. From these results, it is clear that the unstrained outer solution does not match the inner solution.

Note that the entropy layer becomes thinner as $\kappa \to 1$. This is related to the fact that there is an entropy-layer effect for $\kappa \to 1$ and $\epsilon = 0(1)$ which would occur in the supersonic flow over a circular cone at small yaw.

References

[1] Hayes, W. D. and Probstein, R. F., *Hypersonic Flow Theory* (Academic Press, New York, 1959), pp. 125–127.

[2] Melnik, R. E. and Scheuing, R. A., "Shock layer structure and entropy layers in hypersonic conical flows," *Hypersonic Flow Research*, edited by F. R. Riddell (Academic Press, New York, 1962).

[3] Cheng, H. K., "Hypersonic shock-layer-theory of a yawed cone and other three dimensional pointed bodies," Wright Air Development Center WADC TN 59-335 (October 1959).

[4] Melnik, R. E., "A conical thin-shock-layer theory uniformly valid in the entropy layer," Flight Dynamics Lab. TDR 64-82 (January 1965).

[5] Scheuing, R. A., Mead, H. R., Brook, J. W., Melnik, R. E., Hayes, W. D., Gray, K. E., Donaldson, C. duP., and Sullivan, R. D., "Theoretical prediction of pressures in hypersonic flow with special reference to configurations having attached leading edge shocks," Aeronautical Systems Div. ASD TR 61-60, part I (May 1962).

[6] Tsien, H. S., "The Poincaré-Lighthill-Kuo method," *Advances in Applied Mechanics* (Academic Press, New York, 1956), Vol. 4.

Radiation Stresses on Real Surfaces

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Nomenclature

F = arbitrary function of θ and φ, Eq. (4)
G = solar irradiation normal to a surface, ergs/sec-cm²
I = spectral radiant intensity, ergs/sec-cm²-sr-μ; I^+ = leaving, I^- = incident, I_b = ideal Planckian radiator
P = surface stress, dynes/cm²-sr-μ; P_x, P_y, and P_z = stresses in the x, y, and z directions of the surface
T = absolute temperature, °K
c = velocity of light, 2.998×10^{10} cm/sec
= reflection distribution function, as defined by $I^+(\theta_2, \varphi_2) = f(\theta_1, \varphi_1, \theta_2, \varphi_2)I^-(\theta_1, \varphi_1) \cos\theta_1 d\Omega_1$ where Ω_1 is the solid angle of the source
h = Planck's const, 6.625×10^{-27} erg-sec
ϵ = emittance
θ = polar angle of spherical coordinate system; θ_1 = incident, θ_2 = leaving, θ_0 = incidence for solar irradiation
λ = wavelength, μ; used as a subscript to denote the monochromatic dependency of the subscripted quantity
ν = frequency, sec^{-1}
π = 3.14159
ρ = reflectance
φ = azimuthal angle of spherical coordinate system; φ_1 = incident, φ_2 = leaving, φ_0 = incidence for solar irradiation

RECENTLY there have been a number of papers on radiation forces. Clancy and Mitchell[1] have shown the effect of radiation-induced torques upon satellite attitude under the assumption of specularly reflecting surfaces. A similar analysis was performed by Polyakhova with the specular assumption. Holl[3] utilized Fresnel's equations as a more realistic approach to the angular variation of reflectance. The purpose of this note is to give the general expression for the radiation stress on a surface of arbitrary reflecting properties, i.e., real, not necessarily perfectly specular or perfectly diffuse. Knowledge of these stresses is important in accurately predicting disturbance torques on gravity-gradient stabilized systems.

In the computation of radiation transfer between real surfaces, spectral radiant intensity $I(\theta, \varphi, \lambda)$ (ergs/sec-cm²-sr-μ) is employed to give the power in direction θ, φ per unit normal area, solid angle, and wavelength bandwidth. The quantity I can be determined when the reflection distribution function is known. These quantities, the radiant intensity and reflection distribution function, can be employed to calculate force per unit area on real surfaces caused by impinging or departing streams of photons. Such surface stresses (not just pressures) are needed in order to calculate forces and moments causing space vehicles to change trajectory and attitude.

According to Einstein's photoelectric law, any particle carries energy $h\nu$. A photon traveling at the speed of light carries momentum $h\nu/c$. The momentum stream per unit normal area, solid angle, and wavelength bandwidth in the direction θ, φ is then I/c (dynes/cm²-sr-μ). This stream gives rise to a force according to Newton's laws. The stress vector on a surface is thus equal to minus the integral of the momentum stream over all solid angles:

$$\begin{vmatrix} P_x \\ P_y \\ P_z \end{vmatrix} = - \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} \begin{vmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{vmatrix} \left(\frac{I}{c}\right) \sin\theta \cos\theta d\theta \varphi d\lambda \tag{1}$$

Two radiant fluxes must be distinguished, one impinging with (Fig. 1) intensity I^- and the other departing with intensity I^+. If the incident flux I^- is known, the departing flux from a surface in a state of thermodynamic equilibrium at temperature T is given by the reflection distribution function[4] $f(\theta_1, \varphi_1, \theta_2, \varphi_2, \lambda)$ and the directional emissivity (related to an integral of the reflection distribution function).[5] The subscript 1 denotes incidence directions, and subscript 2 denotes departing directions:

$$I^+(\theta_2, \varphi_2, \lambda) = \epsilon(\theta_2, \varphi_2)I_b(T, \lambda) + \int_0^{2\pi} \int_0^{\pi/2} I^-(\theta_1, \varphi_1, \lambda)f(\theta_1, \theta_1, \theta_2, \varphi_2, \lambda) \sin\theta_1 \cos\theta_1 d\theta_1 d\varphi_1 \tag{2}$$

where I_b is the blackbody intensity by Planck's law. Thus the complete expression for the surface stress is

$$\begin{vmatrix} P_x \\ P_y \\ P_z \end{vmatrix} = - \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} \begin{vmatrix} \sin\theta_1 \cos\varphi_1 \\ \sin\theta_1 \sin\varphi_1 \\ \cos\theta_1 \end{vmatrix} \times \left(\frac{I^-}{c}\right) \sin\theta_1 \cos\theta_1 d\theta_1 d\varphi_1 d\lambda - \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} \begin{vmatrix} \sin\theta_2 \cos\varphi_2 \\ \sin\theta_2 \sin\varphi_2 \\ \cos\theta_2 \end{vmatrix} \times \left\{ \epsilon(\theta_2, \varphi_2, \lambda)\left[I_b\left(\frac{T\lambda}{c}\right)\right] + \int_0^{2\pi} \int_0^{\pi/2} \left(\frac{I^-}{c}\right) \times f(\theta_1, \varphi_1, \theta_2, \varphi_2, \lambda) \sin\theta_1 \cos\theta_1 d\theta_1 d\varphi_1 \right\} \sin\theta_2 \cos\theta_2 d\theta_2 d\varphi_2 d\lambda \tag{3}$$

Several special cases of Eq. (3) can be readily integrated to closed form. First, if I^- is nearly collimated as is solar radiation at the earth's distance from the sun, the integration over θ_1 and φ_1 reduces as follows:

$$\int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} I^- \sin\theta_1 \cos\theta_1 F(\theta_1,\varphi_1) d\theta_1 d\varphi_1 d\lambda = G \cos\theta_0 F(\theta_0, \varphi_0) \tag{4}$$

Angle θ_0 is the polar angle between the surface normal and the line of sight to the center of the sun or other source, and φ_0 is the azimuthal angle measured in the plane of the surface. Taking F to be successively each of the functions in Eq. (3) transforms Eq. (3) to

$$\begin{vmatrix} P_x \\ P_y \\ P_z \end{vmatrix} = - \begin{vmatrix} \sin\theta \cos\varphi_0 \\ \sin\theta_0 \sin\varphi_0 \\ \cos\theta_0 \end{vmatrix} \left(\frac{G}{c}\right) \cos\theta_0 - \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} \left\{ \begin{vmatrix} \sin\theta_2 \cos\varphi_3 \\ \sin\theta_2 \cos\varphi_2 \\ \cos\theta_2 \end{vmatrix} \epsilon(\theta_2, \varphi_2, \lambda)\left[I_b\left(\frac{T\lambda}{c}\right)\right] + \left(\frac{G_\lambda}{c}\right) \cos\theta_0 f(\theta_0, \varphi_0, \theta_2, \varphi_2, \lambda) \right\} \sin\theta_2 \cos\theta_2 d\theta_2 d\varphi_2 d\lambda \tag{5}$$

If the reflection distribution function is that for a perfectly diffuse or a perfectly specular surface, then Eq. (5) can be reduced further. For a perfectly diffuse surface, $f(\theta_0, \varphi_0, \theta_0, \varphi_2, \lambda)$

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